T.Y.B.Sc. : Semester - VI

US06CMTH24

RIEMANN INTEGRATION AND SERIES OF FUNCTIONS

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1. Pointwise Convergence

Pointwise Convergence:
Let $\{f_n\}$ be a sequence of real valued functions defined on an interval I. If for each $x \in I$ the limit $\lim\limits_{n\to\infty} f_n(x)$ exists then a function f defined on I by

$$\lim_{n\to\infty} f_n(x) = f(x)$$

is called the limit of $\{f_n\}$ as n tends to ∞ and the sequence of functions $\{f_n\}$ is said to be point-wise convegent to f.

Note: Equivalenly we can define point-wise convergence as follows.

Let $\{f_n\}$ be a sequence of real valued functions defined on an interval I and f also be a function defined on I. If for each $\epsilon > 0$ and each $x \in I$ there exists some positive integer m, depending on choice of x and ϵ such that

$$|f_n(x) - f(x)| < \epsilon, \quad \text{whenever, } n \geqslant m$$

then the sequence of functions $\{f_n\}$ is said to be point-wise convegent to f. Also f is called point-wise limit of $\{f_n\}$ as n tends to ∞ and it is written as,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

2. Uniform Convergence

Unifrom Convergence:

Let $\{f_n\}$ be a sequence of real valued functions defined on an interval I and f also be a function defined on I. If for each $\epsilon > 0$ and **every** $x \in I$ there exists some positive integer m, **independent** of choice of x in I, such that

$$|f_n(x) - f(x)| < \epsilon, \quad \text{whenever, } n \geqslant m$$

then the sequence of functions $\{f_n\}$ is said to be uniformly convegent to f. Also f is called uniform limit of $\{f_n\}$ as n tends to ∞.

3.	**State and prove Cauchy's criteria for uniform convergence of a sequence of functions.**

Cauchy's criteria for uniform convergence of a sequence of functions:
A sequence of functions $\{f_n\}$ defined on $[a, b]$ converges uniformly in $[a, b]$ on $[a, b]$ if and only if every $\epsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that,

$$|f_{n+p}(x) - f_n(x)| < \epsilon, \ \ \forall n \geqslant N, \ p \geqslant 1$$

Proof:
First, suppose $\{f_n\}$ of functions converges uniformly on $[a, b]$ to the limit function f.

Then, for any given $\epsilon > 0$ and every choice of $x \in [a, b]$, there exists some positive integer N, independent of x, such that,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \quad \text{whenever, } n \geqslant N$$

For every $p \geqslant 1$ since $n + p > N$ we have,

$$|f_{n+p}(x) - f(x)| < \frac{\epsilon}{2}$$

Therefore, for $n \geqslant N$ and $p \geqslant 1$, we have

$$\begin{aligned} |f_{n+p}(x) - f_n(x)| &= |f_{n+p}(x) - f(x) + f(x) - f_n(x)| \\ &\leqslant |f_{n+p}(x) - f(x)| + |f(x) - f_n(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ \therefore, |f_{n+p}(x) - f_n(x)| \epsilon, & \quad \text{whenever } n \geqslant N, \ p \geqslant 1 \end{aligned}$$

Conversely, suppose for any given $\epsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that,

$$|f_{n+p}(x) - f_n(x)| < \epsilon, \ \ \forall n \geqslant N, \ p \geqslant 1 \text{ - - - } (1)$$

Therefore, By Cauchy's general princial for convergence, for each $x \in [a, b]$, sequence of real numbers $\{f_n(x)\}$ converges to a limit, say $f(x)$. Therefore, $\{f_n(x)\}$ converges point-wise to f.

If we fix any n in (1) and let $p \to \infty$ then we have $f_{n+p} \to f$

Therefore, we have

$$|f_n(x) - f(x)| < \epsilon, \quad \text{whenever, } n \geqslant N \ \forall x \in [a, b]$$

Hence, $\{f_n(x)\}$ conveges uniformly to f.

> 4. **Let $\{f_n\}$ be a sequence of functions such that $\lim_{n\to\infty} f_n(x) = f(x),\ x \in [a, b]$ and let**
> $$M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$$
> **Then $f_n \to f$ uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.**

Proof:
Let $f_n \to f$ uniformly on $[a, b]$.

Therefore, for a given $\epsilon > 0$ there exists a positive integer N such that,

$$|f_n(x) - f(x)| < \epsilon,\ \forall\ n \geqslant N,\ \forall x \in [a, b]$$

Since $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$, we have,

$$Mn < \epsilon,\ \forall n \geqslant N$$

Therefore

$$M_n \to 0 \text{ as } n \to \infty$$

Conversely, suppose, $M_n \to 0$ as $n \to \infty$. Therefore, for any $\epsilon > 0$ there exists a postitive integer N such that,

$$M_n < \epsilon,\ \forall n \geqslant N$$

Therefore,

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| < \epsilon,\ \forall\ n \geqslant N$$

Therefore,

$$|f_n(x) - f(x)| < \epsilon,\ \forall\ n \geqslant N,\ \forall x \in [a, b]$$

Hence, $f_n \to f$ uniformly on $[a, b]$.

> 5. **Uniform convergence of series of functions.**

Uniform convergence of series of functions.
A series $\sum f_n$ of functions is said to converge uniformly on an inverval $[a, b]$ if the sequence $\{S_n\}$ of its partial sums defined by

$$S_n = \sum_{i=1}^{n} f_n(x)$$

converges uniformly on $[a, b]$.

> 6. **State and prove Weistrass' s M-test.**

Weistrass' s M-test

A series $\sum f_n$ of functions is uniformly (and absolutely) convergent on $[a, b]$ if there exists a convergent series $\sum M_n$ of postive numbers such that for all $x \in [a, b]$,

$$|f_n(x)| \leqslant M_n, \; \forall n$$

Proof:

Let $\sum M_n$ be a convergent series of positive numbers such that,

$$|f_n(x)| \leqslant M_n, \; \forall n$$

By Cauchy's criteria convergence of series of numbers, $\sum M_n$ is convergent iff for a given $\epsilon > 0$ there exists some postitve integer N such that,

$$|M_{n+1} + M_{n+2} + \cdots + M_{n+p}| < \epsilon, \; \forall n \geqslant N, \; p \geqslant 1$$

Now,

$$\begin{aligned} |f_{n+1}(x) + f_{n+1}(x) + \cdots + f_{n+p}(x)| &\leqslant |f_{n+1}(x)| + |f_{n+1}(x)| + \cdots + |f_{n+p}(x)| \\ &\leqslant M_{n+1} + M_{n+2} + \cdots + M_{n+p} \\ \therefore |f_{n+1}(x) + f_{n+1}(x) + \cdots + f_{n+p}(x)| &< \epsilon, \; \forall n \geqslant N, \; p \geqslant 1 \end{aligned}$$

Hence, series $\sum f_n$ of functions is uniformly (and absolutely) convergent on $[a, b]$.

7. **State and prove Abel's test.**

Abel's test.

If $b_n(x)$ is a positive monotonic decreasing function of n for each fixed value of x in the interval $[a, b]$ and $b_n(x)$ is bounded for all values of n and x concerned, and if the series $\sum u_n(x)$ is uniformly convergent on $[a, b]$, then so also is the serie $\sum b_n(x)u_n(x)$.

Proof:

Since, $b_n(x)$ is bounded for all values of n and x, there exists a postive number K, independent of x and n, such that, for all $x \in [a, b]$ and $n = 1, 2, \ldots$

$$0 \leqslant b_n(x) \leqslant K$$

Now, if $\sum u_n(x)$ is a uniformly convergent series then for a given $\epsilon > 0$ there exists some positive integer N such that,

$$\sum_{r=n+1}^{n+p} u_r(x) < \frac{\epsilon}{K}, \; \forall n \geqslant N, \; p \geqslant 1$$

Hence, by Abel's lemma, we get

$$\sum_{r=n+1}^{n+p} b_r(x).u_r(x) \leqslant b_{n+1}(x) \quad \max_{q=1,2,\ldots,p} \left| \sum_{r=n+1}^{n+q} u_r(x) \right|$$

$$< K\frac{\epsilon}{K},\ \forall n \geqslant N,\ p \geqslant 1,\ x \in [a,b]$$

$$\therefore, \sum_{r=n+1}^{n+p} u_r(x) < \epsilon,\ \ \forall n \geqslant N,\ p \geqslant 1,\ x \in [a,b]$$

Hence, $\sum b_n(x).u_n(x)$ is uniformly convergent on $[a,b]$

8. State and prove Dirichlet's test.

Dirichlet's test

If $b_n(x)$ is a monotonic function of n for each fixed value of x in the interval $[a,b]$ and $b_n(x)$ tends uniformly to zero for $a \leqslant x \leqslant b$, and if there is a number $K > 0$ independent of x and n, such that for all values of x in $[a,b]$,

$$\left|\sum_{r=1}^{n} u_r(x)\right| \leqslant K,\ \forall n$$

then the series $\sum b_n(x)u_n(x)$ is uniformly convergent on $[a,b]$.

Proof:

As $b_n(x)$ converges uniformly to 0, and for any $\epsilon > 0$ there exists some positive integer N, independent of x, such that,

$$|b_n(x)| \leqslant \frac{\epsilon}{4K},\ \ \forall n \geqslant N$$

Let $S_n = \sum\limits_{r=1}^{n} u_r(x)$. Therefore,

$$|S_n| \leqslant K,\ \ \forall n,\ \forall x \in [a,b]$$

Now,

$$\sum_{r=n+1}^{n+p} b_r(x).u_r(x) = b_{n+1}(x)u_{n+1}(x) + b_{n+2}(x)u_{n+2}(x) + \cdots + b_{n+p}(x)u_{n+p}(x)$$

$$= b_{n+1}(x)\left(S_{n+1} - S_n\right) + b_{n+2}(x)\left(S_{n+2} - S_{n+1}\right) + \cdots + b_{n+p}(x)\left(S_{n+p} - S_n\right)$$

$$= -b_{n+1}(x)S_n + (b_{n+1}(x) - b_{n+2}(x))S_{n+1} + (b_{n+2}(x) - b_{n+3}(x))S_{n+2} + \ldots$$
$$+ (b_{n+p-1}(x) - b_{n+p}(x))S_{n+p-1} + b_{n+p}(x)S_{n+p}$$

$$= \sum_{r=n+1}^{n+p-1} (b_r(x) - b_{r+1}(x))S_r - b_{n+1}(x)S_n + b_{n+p}(x)S_{n+p}$$

$$\therefore\ \left|\sum_{r=n+1}^{n+p} b_r(x).u_r(x)\right| \leqslant K\left(\sum_{r=n+1}^{n+p-1} |b_r(x) - b_{r+1}(x)| + |b_{n+1}(x)| + |b_{n+p}(x)|\right) \quad (\because |S_n| \leqslant K)$$

$$= K\left(|b_{n+1}(x) - b_{n+p}(x)| + |b_{n+1}(x)| + |b_{n+p}(x)|\right) \quad (\because b_n(x) \text{ is monotonic.})$$

$$\leqslant K\left(|b_{n+1}(x)| + |b_{n+p}(x)| + |b_{n+1}(x)| + |b_{n+p}(x)|\right)$$

$$= K\left(\frac{\epsilon}{4K} + \frac{\epsilon}{4K} + \frac{\epsilon}{4K} + \frac{\epsilon}{4K}\right)$$

$$< K\left(\frac{\epsilon}{4K}\right), \ \forall n \geqslant N, \ p \geqslant 1, \ x \in [a, b]$$

$$\therefore \quad \left|\sum_{r=n+1}^{n+p} b_r(x)u_r(x)\right| < \epsilon, \quad \forall n \geqslant N, \ p \geqslant 1, \ x \in [a, b]$$

Hence, by Cauchy's criteria $\sum b_n(x).u_n(x)$ is uniformly convergent on $[a, b]$.